Scattering theory in a time-dependent external field. II. Applications



1974 J. Phys. A: Math. Nucl. Gen. 7 586

(http://iopscience.iop.org/0301-0015/7/5/008)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 5, 1974. Printed in Great Britain. © 1974.

# Scattering theory in a time-dependent external field II. Applications

E Prugovečki† and A Tip‡

† Department of Mathematics, University of Toronto, Toronto M5S 1A1, Canada
‡ FOM-Instituut voor Atoom- en Molecuulfysica, Kruislaan 407, Amsterdam/Wgm, The Netherlands

Received 25 September 1973

**Abstract.** The general theory of the scattering of two particles in a time-dependent external field, presented in paper I, is applied to the case of two point particles with internal structure which interact through a potential with matrix representation $\{V_{kl}(r)\}$. Under suitable, quite general, conditions upon the $l^2$ operator bound $|V(r)|$ of $\{V_{kl}(r)\}$ the existence of the wave operators $\Omega_\pm(s)$ is proven for two field configurations. The first is that of a spatially homogeneous time-dependent field, whereas in the second case the field is inhomogeneous but localized in space. Some attention is paid to the problem of the existence of differential cross sections and their relation to the scattering operator $S(s) = \Omega_+^*(s)\Omega_-(s)$.

## 1. Introduction

In the present paper we continue the discussion of the quantum-mechanical scattering of particles in the presence of a time-dependent external field (Prugovečki and Tip 1974, to be referred to as I). References to formulae and theorems of paper I are given in the form (I-2.3) and theorem I-3.1, for example. Here we will apply the general formalism set up in I to the case of two neutral point particles with internal structure which collide with each other in the presence of an external field. The model we use is the one discussed in I, § 1. We shall consider two special forms of the external field contribution to the hamiltonian. In § 2 we prove the existence of the wave operators under suitable assumptions on the interaction potential for the case of a spatially homogeneous time-dependent external field, whereas in § 3 we do so for the situation where the field is spatially inhomogeneous but sufficiently localized in space. The paper ends with a discussion section. There we pay some attention to the question whether quantities such as cross sections do exist in the present case.

## 2. Spatially homogeneous time-dependent external fields

In the present section we consider the special case of a system consisting of two particles with internal structure which is subjected to a time-dependent external field, acting solely on the internal degrees of freedom of the particles (this model was introduced in § I.1). Hence we can eliminate from the outset the centre-of-mass motion of the system and work in a Hilbert space $\mathscr{H} = \mathscr{H}^{\text{rel}} \otimes \mathscr{H}^{\text{int}}$, where $\mathscr{H}^{\text{rel}} = L^2(\mathscr{R}^3)$ refers to the relative motion of the two particles and $\mathscr{H}^{\text{int}} = l^2(n)$ for fixed $n \leqslant \infty$ refers to their internal structure. (In fact $\mathscr{H}^{\text{int}}$ is the direct product of the two internal structure Hilbert spaces for each particle separately but this fact does not play an explicit role in the present section.) We shall denote the relative position vector of the two particles by $\boldsymbol{r}$, their relative momentum by $\boldsymbol{k}$ and their reduced mass by $m$. The unperturbed hamiltonian reads

$$H_0(t) = K^{\mathrm{rel}} \otimes I^{\mathrm{int}} + I^{\mathrm{rel}} \otimes K^{\mathrm{int}} + I^{\mathrm{rel}} \otimes H^{\mathrm{ext}}(t), \tag{2.1}$$

where $I^{\mathrm{rel}}$ and $I^{\mathrm{int}}$ are the identity operators on $\mathscr{H}^{\mathrm{rel}}$ and $\mathscr{H}^{\mathrm{int}}$ respectively, $K^{\mathrm{rel}} = \boldsymbol{k}^2/2m$, $K^{\mathrm{int}}$ is the internal energy, which in matrix notation has the form

$$K^{\mathrm{int}} = \begin{pmatrix} \omega_1 & & 0 \\ & \omega_2 & \\ 0 & & \ddots \end{pmatrix}, \tag{2.2}$$

whereas $H^{\mathrm{ext}}(t)$ is the external field contribution to the internal hamiltonian. We assume that $H^{\mathrm{ext}}(t)$ is a bounded operator on $\mathscr{H}^{\mathrm{int}} = l^2(n)$ and that the conditions (i)–(iii) of I, § 2, are satisfied. Since the relative and internal parts of $H_0(t)$ commute, the evolution operator $U_0(t, s)$, describing the motion of the two non-interacting particles in the external field, factorizes in the following manner:

$$U_0(t, s) = \exp[-\mathrm{i}K^{\mathrm{rel}}(t-s)] \otimes U_0^{\mathrm{int}}(t, s), \tag{2.3}$$

where $U_0^{\mathrm{int}}(t, s)$ is the internal motion evolution operator, acting on $\mathscr{H}^{\mathrm{int}}$, whose infinitesimal generator at $t = s$ is $K^{\mathrm{int}} + H^{\mathrm{ext}}(t)$.

The two particles are assumed to interact through potentials which depend on the internal states of the particles. Therefore the self-adjoint operator $V$ on $\mathscr{H}$ has the general form

$$V(\boldsymbol{r})\begin{pmatrix} \psi_1(\boldsymbol{r}) \\ \psi_2(\boldsymbol{r}) \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} V_{11}(\boldsymbol{r}) & V_{12}(\boldsymbol{r}) \dots \\ V_{21}(\boldsymbol{r}) & V_{22}(\boldsymbol{r}) \dots \\ \vdots & \vdots \ \ddots \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} \psi_1(\boldsymbol{r}) \\ \psi_2(\boldsymbol{r}) \\ \vdots \\ \vdots \end{pmatrix}, \tag{2.4}$$

when acting on an element $\Psi \in \mathscr{D}_V \subset \mathscr{H}$, where $\mathscr{D}_V$ is the domain of definition of $V$. For a fixed value of $\boldsymbol{r}$ we may consider $V(\boldsymbol{r})$ as being an operator on $\mathscr{H}^{\mathrm{int}} = l^2(n)$, which we require to be bounded, ie for $\alpha \in \mathscr{H}^{\mathrm{int}}$

$$\|V(\boldsymbol{r})\alpha\|_{\mathrm{int}} = \left(\sum_i \left|\sum_j V_{ij}(\boldsymbol{r})\alpha_j\right|^2\right)^{1/2} \leqslant |V(\boldsymbol{r})|_{\mathrm{int}}\|\alpha\|_{\mathrm{int}}, \tag{2.5}$$

where $|\dots|_{\mathrm{int}}$ denotes the operator bound on $\mathscr{H}^{\mathrm{int}}$ and $\|\dots\|_{\mathrm{int}}$ the vector norm $(\Sigma_j|\alpha_j|^2)^{1/2} = \|\alpha\|_{\mathrm{int}}$ on $l^2(n)$. Naturally, $|V(\boldsymbol{r})|_{\mathrm{int}}$ depends in general on $\boldsymbol{r} \in \mathscr{R}^3$ and we require that it be locally square integrable and also that

$$|V(\boldsymbol{r})|_{\mathrm{int}} = \mathrm{O}(r^{-1-\epsilon_0}) \tag{2.6}$$

for some fixed $\epsilon_0 > 0$. The full hamiltonian of the system is

$$H(t) = H_0(t) + V. \tag{2.7}$$

We note that, due to the restrictions imposed upon $V$, this hamiltonian is self-adjoint and can be written in the form (I-2.1). Moreover, it satisfies the conditions (i)–(iii) of I, § 2. Consequently we can apply the results of that section to infer the existence of the time evolution operator $U(t, s)$, having the properties stated in theorems I-2.1 and I-2.2. In order to prove the existence of the wave operators (I-3.1) we need the following result.

*Lemma 2.1.* Let $\mathscr{D}_0^{\rm rel}$ denote the fundamental set

$$\mathscr{D}_0^{\rm rel} = \{\psi_\rho(\boldsymbol{r}) | \tilde{\psi}_\rho(\boldsymbol{k}) = k_1 k_2 k_3 \exp[-\boldsymbol{k}^2/(2m) - {\rm i}\boldsymbol{k} \cdot \boldsymbol{\rho}], \boldsymbol{\rho} \in \mathscr{R}^3\} \tag{2.8}$$

of vectors in $L^2(\mathscr{R}^3)$, where $\tilde{\psi}(\boldsymbol{k})$ is the Fourier transform of the function $\psi(\boldsymbol{r})$ (a fundamental set in a linear topological space is a set whose linear span is dense in that space). If $V$ satisfies the above conditions then for any element $\Psi \in \mathscr{D}_0^{\rm rel} \otimes_{\rm a} \mathscr{H}^{\rm int}$, the algebraic tensor product of $\mathscr{D}_0^{\rm rel}$ and $\mathscr{H}^{\rm int}$, there is a constant $C_\Psi$ depending on $\Psi$ and a fixed number $\delta > 0$ so that

$$\| V U_0(t, s)\Psi \| \leqslant C_\Psi |1 + {\rm i}(s-t)|^{-1-\delta} \tag{2.9}$$

for all $s, t \in \mathscr{R}$.

*Proof.* Any $\Psi \in \mathscr{D}_0^{\rm rel} \times \mathscr{H}^{\rm int} \subset \mathscr{D}_0^{\rm rel} \otimes_{\rm a} \mathscr{H}^{\rm int}$ is of the form

$$\Psi_\rho(\boldsymbol{r}) = \psi_\rho(\boldsymbol{r}) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix} \tag{2.10}$$

and consequently

$$U_0(t, s)\Psi_\rho(\boldsymbol{r}) = \psi_\rho(\boldsymbol{r}, t-s) \begin{pmatrix} \alpha_1(t, s) \\ \alpha_2(t, s) \\ \vdots \end{pmatrix}, \tag{2.11}$$

where $\psi_\rho(\boldsymbol{r}, t-s) = \exp[-{\rm i}K^{\rm rel}(t-s)]\psi_\rho(\boldsymbol{r})$ and

$$\begin{pmatrix} \alpha_1(t, s) \\ \alpha_2(t, s) \\ \vdots \\ \vdots \end{pmatrix} = U_0^{\rm int}(t, s) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \end{pmatrix}. \tag{2.12}$$

To establish that $VU_0(t, s)\Psi_\rho$ actually belongs to $\mathscr{H} = \mathscr{H}^{\rm rel} \otimes \mathscr{H}^{\rm int}$, note that

$$\sum_i |(VU_0(t, s)\Psi_\rho)_i(\boldsymbol{r})|^2 = \sum_i \Big| \sum_j V_{ij}(\boldsymbol{r})\psi_\rho(\boldsymbol{r}, t-s)\alpha_j(t, s) \Big|^2$$

$$= |\psi_\rho(\boldsymbol{r}, t-s)|^2 \| V(\boldsymbol{r}) U_0^{\rm int}(t, s)\alpha \|^2_{\rm int} \leqslant |\psi_\rho(\boldsymbol{r}, t-s)|^2 |V(\boldsymbol{r})|^2_{\rm int} \|\alpha\|^2_{\rm int},$$

where the unitarity of $U_0^{\rm int}(t, s)$ has been used to arrive at the final expression. Consequently, by Lebesgue's bounded convergence theorem, we conclude that

$$VU(t, s)\Psi_\rho \in L^2(\mathscr{R}^3) \otimes l^2(n) = \mathscr{H}$$

and furthermore

$$\| VU_0(t, s)\Psi_\rho \|^2 = \sum_i \int_{\mathscr{R}^3} {\rm d}\boldsymbol{r} |(VU_0(t, s)\Psi_\rho)_i(\boldsymbol{r})|^2 \leqslant \|\alpha\|^2_{\rm int} \int_{\mathscr{R}^3} {\rm d}\boldsymbol{r} |V(\boldsymbol{r})|^2_{\rm int} |\psi_\rho(\boldsymbol{r}, t-s)|^2. \tag{2.13}$$

Thus we have reduced the problem to a form where the standard methods developed for the case of structureless particles can be applied (cf Prugovečki 1971, pp 541–3), thus arriving at the estimate

$$\left(\int_{\mathscr{R}^3} \mathrm{d}\boldsymbol{r} |V(\boldsymbol{r})|^2_{\mathrm{int}} |\psi_\rho(\boldsymbol{r}, t-s)|^2\right)^{1/2} \leqslant C_\rho |1+\mathrm{i}(t-s)|^{-1-\delta},$$

where $\delta > 0$ is fixed for a given interaction $V$, while $C_\rho$ is a constant that may depend on the parameter $\rho \in \mathscr{R}^3$. This establishes the result for an arbitrary element of the cartesian product $\mathscr{D}_0^{\mathrm{rel}} \times \mathscr{H}^{\mathrm{int}}$. Since any element of the algebraic tensor product $\mathscr{D}_0^{\mathrm{rel}} \otimes_{\mathrm{a}} \mathscr{H}^{\mathrm{int}}$ is a finite linear combination of elements of $\mathscr{D}_0^{\mathrm{rel}} \times \mathscr{H}^{\mathrm{int}}$ the result extends in a simple manner to this more general case and the proof of the lemma is completed.

*Theorem 2.1.* If $|V(\boldsymbol{r})|_{\mathrm{int}}$ is Lebesgue locally square-integrable in $\boldsymbol{r}$ and satisfies (2.6) for some $\epsilon_0 > 0$ then the wave operators $\Omega_\pm(s)$ defined in (I-3.1) exist for any $s \in \mathscr{R}$.

*Proof.* According to theorem I-2.1 we have $U_0(t,s)\Psi \in \mathscr{D}_{H_0(t)}$ for any $\Psi \in \mathscr{D}_{H_0(t)}$. In our case $\mathscr{D}_{H(t)} = \mathscr{D}_{H_0(t)}$ due to the fact that $V$ is relatively bounded with respect to $H_0(t)$ (cf Prugovečki 1971, p 365). Since $\mathscr{D}_0^{\mathrm{rel}} \otimes_{\mathrm{a}} \mathscr{H}^{\mathrm{int}} \subset \mathscr{D}_{H_0(t)}$ and by lemma 2.1

$$\pm\int_1^{\pm\infty} \|VU_0(t,s)\Psi\|\,\mathrm{d}t \leqslant C_\Psi \left|\int_1^{\pm\infty} \mathrm{d}t |1+\mathrm{i}(t-s)|^{-1-\delta}\right| < \infty$$

for any $\Psi \in \mathscr{D}_0^{\mathrm{rel}} \otimes_{\mathrm{a}} \mathscr{H}^{\mathrm{int}}$, where $\mathscr{D}_0^{\mathrm{rel}} \otimes_{\mathrm{a}} \mathscr{H}^{\mathrm{int}}$ is dense in $\mathscr{H}$, we conclude that theorem I-3.2 is applicable with the sets $\mathscr{D}_s \supset \mathscr{D}_0^{\mathrm{rel}} \otimes_{\mathrm{a}} \mathscr{H}^{\mathrm{int}}$ being dense in $\mathscr{H}$ for any $s \in \mathscr{R}$. Thus the existence of $\Omega_\pm(s)$ is established for all $s \in \mathscr{R}$.

We note that in the case $n < \infty$ the conditions imposed upon $V$ are satisfied if each component $V_{ij}(\boldsymbol{r}) = \overline{V_{ji}}(\boldsymbol{r})$ satisfies these same conditions, ie it is locally square-integrable and vanishes at infinity faster than $r^{-1-\epsilon_{ij}}$ with $\epsilon_{ij} > 0$.

Thus theorem 2.1 does not apply to potentials which behave asymptotically as Coulomb potentials $V_{ij}(\boldsymbol{r})$ or other long-range potentials. However, these cases can be easily included if 'renormalized' wave operators are introduced (see, for instance, Prugovečki and Zorbas 1973). On the other hand, if $n = \infty$ the theorem requires a fast decrease in the interaction strength between the highly excited internal energy levels so as to ensure the existence of $|V(\boldsymbol{r})|_{\mathrm{int}}$ for any $\boldsymbol{r} \in \mathscr{R}$.

It should also be pointed out that the present results for neutral particles with internal structure can be extended to the case when three or more such particles are simultaneously involved in the collision process. This generalization is especially straightforward when $V$ contains only two-body potentials (cf Prugovečki 1971, p 588) since then the entire method of proving the existence of channel wave operators is essentially the same as in the above mentioned two-body case. This result is relevant if one attempts to construct a kinetic theory for a gas of particles with internal structure in a radiation field, since it guarantees the existence of the long-time limits of $n$ particle streaming operators (see also I, § 1).

## 3. Spatially localized time-dependent fields

The main problem considered in this section concerns the case of two particles with internal structure colliding in the presence of an external field which is localized in space and hence depends on position as well as on time. This type of problem obviously does not possess translational invariance since the influence of the external field will now also depend on the position of the centre-of-mass of the two particles. Consequently, the centre-of-mass motion cannot be separated from the beginning, as was done in the preceding section.

Let us denote by $\mathscr{H}_j$ the Hilbert space of the $j$th particle when it is isolated and constitutes an independent system. We assume that $\mathscr{H}_j = L^2(\mathscr{R}^3) \otimes l^2(n_j)$, where $n_j \leqslant \infty$ is the number of internal states of the $j$th particle. The Hilbert space $\mathscr{H}$ of the two-particle system is now $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$. The free hamiltonian $H_0$ is taken to be the sum of the hamiltonians of the two free particles, so that

$$H_0 = K_1 \otimes I_2 + I_1 \otimes K_2 \tag{3.1}$$

where $K_j$ consists of a translational and an internal part, ie $K_j = K_j^{\text{tr}} + K_j^{\text{int}}$, and $I_j$ is the identity operator on $\mathscr{H}_j$. In the sequel we shall denote by $I_j^{\text{tr}}$ and $I_j^{\text{int}}$ the identity operators on the Hilbert space associated with the translational and internal motion of the $j$th particle, respectively. The total hamiltonian $H(t)$ has the form

$$H(t) = H_0 + V + H^{\text{ext}}(t), \tag{3.2}$$

where $V$ is described by a matrix on $l^2(n_1) \otimes l^2(n_2)$

$$V(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \begin{pmatrix} V_{11,11}(\boldsymbol{x}_1 - \boldsymbol{x}_2) & V_{11,12}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \dots \\ V_{12,11}(\boldsymbol{x}_1 - \boldsymbol{x}_2) & V_{12,12}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \dots \end{pmatrix}, \tag{3.3}$$

whose matrix elements depend on the relative position vector of the two particles. The term $H^{\text{ext}}(t)$ describes the influence of the external field on the system and is assumed to be of the form

$$\begin{aligned} H^{\text{ext}}(t) &= H_1^{\text{ext}}(t) \otimes I_2 + I_1 \otimes H_2^{\text{ext}}(t), \\ H_j^{\text{ext}}(t) &= \gamma(\boldsymbol{x}_j, t) I_j^{\text{tr}} \otimes h_j^{\text{ext}} \end{aligned} \tag{3.4}$$

(see (I-1.5) for an example).

We shall assume throughout this section that the operator bound $|V(\boldsymbol{r})|_{\text{int}}$ of $V(\boldsymbol{r})$ in $l^2(n_1) \otimes l^2(n_2)$ is a locally square-integrable function in $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2 \in \mathscr{R}^3$ and that

$$|V(\boldsymbol{r})|_{\text{int}} = \mathrm{O}(r^{-1-\epsilon_0}) \tag{3.5}$$

for some $\epsilon_0 > 0$. In addition we shall also assume that $h_j^{\text{ext}}$ is a bounded operator on $l^2(n_j)$ and that $|\gamma(\boldsymbol{x}, t)|$ is majorized by a time-independent locally square-integrable function $\Gamma(\boldsymbol{x})$ for which

$$\Gamma(\boldsymbol{x}) = \mathrm{O}(|\boldsymbol{x}|^{-1-\epsilon_1}) \tag{3.6}$$

for some $\epsilon_1 > 0$ ($\gamma(\boldsymbol{x}, t)$ is essentially the amplitude of the external field). Under these conditions $H(t)$ obviously satisfies the conditions (i)–(iii) of I, § 2. Consequently, we can state that $H_0$ and $H(t)$ have corresponding time-evolution operators

$$U_0(t-s) = \exp[-\mathrm{i}H_0(t-s)]$$

and $U(t, s)$, respectively, as specified in theorems I-2.1 and I-2.2.

*Lemma 3.1.* Let $\mathscr{D}_j$ denote the fundamental set

$$\{\psi_\rho(\boldsymbol{x})|\tilde{\psi}_\rho(\boldsymbol{k}) = \exp[-\boldsymbol{k}^2/(2m_j) - i\boldsymbol{k}\,.\,\rho], \rho \in \mathscr{R}^3\} \tag{3.7}$$

in $\mathscr{H}_j^{\text{tr}}(= L^2(\mathscr{R}^3))$, the Hilbert space associated with the translational motion of particle $j$. If $H^{\text{ext}}(t)$ is as specified above, then for any element $\Psi$ of the algebraic tensor product $\mathscr{D} = \mathscr{D}_1 \otimes_a l^2(n_1) \otimes_a \mathscr{D}_2 \otimes_a l^2(n_2)$ there is a constant $C_\Psi^{(1)}$, depending only on $\Psi$, so that

$$\|H^{\text{ext}}(t)U_0(t-s)\Psi\| \leqslant C_\Psi^{(1)}[1+4(t-s)^2]^{-\frac{1}{4}-\epsilon_1/4} \tag{3.8}$$

for all $s, t \in \mathscr{R}$.

*Proof.* We shall only consider the case when $\Psi$ has the form

$$\Psi = \psi_1 \otimes \alpha_1 \otimes \psi_2 \otimes \alpha_2, \qquad \psi_j \in \mathscr{D}_j, \qquad \alpha_j \in l^2(n_j), \tag{3.9}$$

since the more general case corresponds to a finite linear combination of expressions of this type. In view of the form (3.4) of $H^{\text{ext}}(t)$, we have in obvious notation

$$\begin{aligned}
&\|H^{\text{ext}}(t)U_0(t-s)\Psi\| \\
&\quad \leqslant \|H_1^{\text{ext}}(t)\exp[-iK_1^{\text{tr}}(t-s)]\psi_1 \otimes \exp[-iK_1^{\text{int}}(t-s)]\alpha_1\|_1\|\psi_2 \otimes \alpha_2\|_2 \\
&\quad\quad +1 \leftrightarrow 2 \\
&\quad \leqslant |h_1^{\text{ext}}|_{1,\text{int}}\|\alpha_1\|_{1,\text{int}}\|\psi_2 \otimes \alpha_2\|_2 \left(\int_{\mathscr{R}^3} d\boldsymbol{x}\Gamma^2(\boldsymbol{x})|\exp[-iK_1^{\text{tr}}(t-s)]\psi_{\rho_1}(\boldsymbol{x})|^2\right)^{1/2} \\
&\quad\quad +1 \leftrightarrow 2,
\end{aligned}$$

where $\|\ldots\|_{j,\text{int}}$ and $|\ldots|_{j,\text{int}}$ denote the norm and operator bound on $l^2(n_j)$. The desired result follows now from standard estimates of the above intergrals which are valid under the conditions imposed on $\Gamma(\boldsymbol{x})$ (cf Kato 1966, pp 534, 535).

*Lemma 3.2.* Suppose $V$ is as specified at the beginning of this section. Then for any $\Psi$ in the domain $\mathscr{D}$ defined in lemma 3.1 there is a constant $C_\Psi^{(2)}$ such that

$$\|VU_0(t-s)\Psi\| \leqslant C_\Psi^{(2)}[1+4(t-s)^2]^{-\frac{1}{4}-\epsilon_0/4} \tag{3.10}$$

for all $s, t \in \mathscr{R}$.

*Proof.* It is obviously again sufficient to prove the lemma for elements $\Psi$ of $\mathscr{D}$ which have the form (3.9). However, since each element of the matrix (3.3), which defines $V$, is a function of the relative position vector only, we take advantage of this fact and recast $\tilde{\psi}_{\rho_1}(\boldsymbol{k}_1)\tilde{\psi}_{\rho_2}(\boldsymbol{k}_2)$, in the form $\tilde{\psi}_{\rho'}^{\text{cm}}(\boldsymbol{K})\tilde{\psi}_{\rho''}^{\text{rel}}(\boldsymbol{k})$, where $\boldsymbol{K}$ is the centre-of-mass momentum, $\boldsymbol{k} = \boldsymbol{k}_1 - \boldsymbol{k}_2$ and $\tilde{\psi}_{\rho'}^{\text{cm}}(\boldsymbol{K}) = \exp[-K^2/(2M) - i\rho'\,.\,\boldsymbol{K}]$, $\tilde{\psi}_{\rho''}^{\text{rel}}(\boldsymbol{k}) = \exp[-k^2/(2m) - i\rho''\,.\,\boldsymbol{k}]$. Here $M = m_1 + m_2$ and $m = m_1m_2/(m_1+m_2)$, whereas $\rho'$ and $\rho''$ are defined through

$$\rho_1 = (m_1/M)\rho' + \rho'', \qquad \rho_2 = (m_2/M)\rho' - \rho''.$$

After this transformation is performed and $H_0$ is written in the form (in self-explanatory notation)

$$H_0 = K^{\text{cm}} \otimes I^{\text{rel}} \otimes I^{\text{int}} + I^{\text{cm}} \otimes K^{\text{rel}} \otimes I^{\text{int}} + I^{\text{cm}} \otimes I^{\text{rel}} \otimes K^{\text{int}},$$

we immediately arrive at the following estimate:

$$\|VU_0(t)\Psi\| = \|\psi^{\rm cm}_{\rho'}\|_{\rm cm}\|V\exp[-{\rm i}(K^{\rm rel}+K^{\rm int})t]\psi^{\rm rel}_{\rho''}\otimes\alpha_1\otimes\alpha_2\|_{\rm rel,int}$$

$$\leqslant \|\psi^{\rm cm}_{\rho'}\|_{\rm cm}\|\alpha_1\|_{1,\rm int}\|\alpha_2\|_{2,\rm int}\left(\int_{\mathscr{R}^3}{\rm d}\boldsymbol{r}|V(\boldsymbol{r})|^2_{\rm int}|(\exp[-{\rm i}K^{\rm rel}t]\psi^{\rm rel}_{\rho''})(\boldsymbol{r})|^2\right)^{1/2}$$

and the rest of the proof follows from the same standard arguments as used at the end of the proof of lemma 3.1.

*Theorem 3.1.* If $V$ and $H^{\rm ext}(t)$ have the properties stated in this section then the wave operators $\Omega_{\pm}(s)$ as defined in (I-3.1) exist for all $s\in\mathscr{R}$.

*Proof.* Since the linear span of each $\mathscr{D}_j$ is dense in $L^2(\mathscr{R}^3)$ by Wiener's theorem (see, for instance, Prugovečki 1971), it follows that $\mathscr{D}$ is dense in $\mathscr{H}$. On the other hand, by lemmas 3.1 and 3.2, for any $\Psi\in\mathscr{D}$

$$\|[V+H^{\rm ext}(t)]U_0(t-s)\Psi\| \leqslant (C^{(1)}_{\Psi}+C^{(2)}_{\Psi})[1+4(t-s)^2]^{-\frac{1}{2}-\epsilon/4},$$

where $\epsilon = \min(\epsilon_0, \epsilon_1)$. This implies that theorem I-3.2 is applicable since $\mathscr{D}\subset\mathscr{D}_{\rm s}$ as a consequence of the above inequality. The wave operators $\Omega_{\pm}(s)$, whose existence is established in the above theorem, refer to the case where the actual motion of the system is compared with the motion of two free particles with no field-influence present. In fact the case considered in this section may, depending on the actual form of $H^{\rm ext}(t)$, give rise to a multi-channel situation. This happens if the field is able to trap one or both particles in a localized space region. (For charged particles this possibility certainly exists, examples being devices such as cyclotrons and storage rings.) Thus there may be open channels referring to the final situation, where, for instance, one particle is trapped and the other asymptotically free. Theorem 3.1 then refers to the channel where both particles become asymptotically free.

## 4. Discussion

In this section we discuss various more of less unrelated aspects of the scattering of particles in a field.

### 4.1. Plane-wave fields

So far we have not considered external fields of the plane-wave type, ie $H^{\rm ext}_j(t)$ in (3.4) of the form

$$H^{\rm ext}_j(t) = \cos(\boldsymbol{k}\cdot\boldsymbol{x}_j-\omega t)I^{\rm tr}_j\otimes h^{\rm ext}_j. \tag{4.1}$$

In this case, where the field is not localized in space, we can no longer compare the true motion of the system with that of two free particles in the absence of a field. Instead we have to consider integrals of the type

$$\int_s^{\infty}{\rm d}t\|VU_0(t,s)\Psi\|, \tag{4.2}$$

where $U_0(t,s)$ is the evolution operator associated with a system, consisting of two non-interacting particles in a field. This situation was encountered in § 2 but in that case the relative translational motion was decoupled from the internal motion. Here this decoupling does not occur, however one might expect that, at least for weak fields, the presence of the time-dependent field leads to some slight modification of the free-particle motion, such that the wave operators still exist. Present day techniques, however, make use of the explicit form of $U_0(t, s)\Psi$, for a conveniently chosen set of $\Psi$. In the present case, due to the complicated form of $U_0(t, s)$, this does not seem to be possible. An alternative might be the expansion of $U_0(t, s)$ in (4.2) in terms of $H^{\text{ext}}(t)$ (see the proof of lemma I-4.2 for this expansion) and to estimate each term separately. As far as we could determine, by considering the first field-dependent term in the expansion, this procedure does not seem to be very promising. If the field does not vary appreciably over the range of the potential, it makes sense to consider a 'long wavelength' approximation, ie $\boldsymbol{x}_j$ in (4.1) is replaced by the centre-of-mass position variable $X$, so that

$$H^{\text{ext}}(t) = \cos(\boldsymbol{k} \,.\, X - \omega t)(I_1^{\text{tr}} \otimes H_1^{\text{ext}} + I_2^{\text{tr}} \otimes H_2^{\text{ext}}). \tag{4.3}$$

In this approximation the relative free-particle motion is decoupled from the centre-of-mass and internal motion and under the usual conditions on the potential (see §§ 2 and 3) the existence of the wave operators can be established.

### 4.2. *High-frequency behaviour*

In I, § 4, we considered the special case of an external field with frequency $\omega$. It is easily verified from the estimates made in §§ 2 and 3 that the conditions of theorem I-4.1 are fulfilled in both cases. In particular the field free wave operators $\Omega_{\pm}^{(\infty)}$ exist and the frequency-dependent wave operators $\Omega_{\pm}^{(\omega)}$ converge strongly to the former for $|\omega| \to \infty$.

### 4.3. *Scattering operator and cross sections*

In analogy with the field-free case we shall say that $\psi \in \mathscr{H}$ is a scattering state if there exist $\psi^{\text{in}}, \psi^{\text{out}} \in \mathscr{H}$, such that $(\psi(t) = U(t, s)\psi)$

$$\lim_{t \to -\infty} \|\psi(t) - U_0(t, s)\psi^{\text{in}}\| = 0,$$
$$\lim_{t \to +\infty} \|\psi(t) - U_0(t, s)\psi^{\text{out}}\| = 0, \tag{4.4}$$

in which case

$$\psi^{\text{out}} = \Omega_+^*(s)\Omega_-(s)\psi^{\text{in}} = S(s)\psi^{\text{in}}. \tag{4.5}$$

In the present case we do not know, whether the scattering operator $S(s)$ is unitary on $\mathscr{H}$, ie whether the collection of all in-states as well as the collection of all out-states coincides with $\mathscr{H}$. The known methods (see, for instance, Prugovečki 1971, § 5) to prove the completeness of the wave operators (ie the relation $\mathscr{R}_{\Omega_+} = \mathscr{R}_{\Omega_-}$ which entails the unitary of $S(s)$) cannot be applied to the present case as the hamiltonian is time dependent, preventing the formulation of a stationary version of the scattering problem. Let us for the moment assume that the completeness problem is solved in the affirmative and see whether cross sections can be defined in terms of $S(s)$. As a preliminary we note that for a given $\psi^{\text{in}} \in \mathscr{H}$ it is easily shown that for any $A \in \mathscr{B}(\mathscr{H})$ which commutes with $U_0(t, s)$

for every $t, s \in \mathscr{R}$

$$\lim_{t \to -\infty} \langle \psi(t) | A \Psi(t) \rangle = \langle \psi^{\text{in}} | A \psi^{\text{in}} \rangle \tag{4.6a}$$

$$\lim_{t \to +\infty} \langle \psi(t) | A \psi(t) \rangle = \langle \psi^{\text{out}} | A \psi^{\text{out}} \rangle$$
$$= \langle S(s)\psi^{\text{in}} | A S(s)\psi^{\text{in}} \rangle = \langle S(0)U_0(0, s)\psi^{\text{in}} | A S(0) U_0(0, s)\psi^{\text{in}} \rangle. \tag{4.6b}$$

Here (I-3.7) and the commutation property of $A$ and $U_0(t, s)$ have been used to arrive at the last equality in (4.6*b*).

Let us consider the situation where asymptotically, for $t \to -\infty$, $\psi(t)$ is confined to a certain subspace $\mathscr{H}_-$ of $\mathscr{H}$, $P_-$ being the projector upon this subspace. We assume that $P_-$ commutes with $U_0(t, s)$ for every real $t, s$. Then, for normalized $\psi^{\text{in}}$ (and hence normalized $\psi(t)$), the above property of $\psi(t)$ can be formulated as

$$\lim_{t \to -\infty} \langle \psi(t) | P_- \psi(t) \rangle = 1. \tag{4.7}$$

Thus, according to (4.6*a*), $\langle \psi^{\text{in}} | P_- \psi^{\text{in}} \rangle = 1$, so that $\psi^{\text{in}} \in \mathscr{H}_-$.

Let $W = W(P_-, \psi^{\text{in}}, P_+)$ be the probability of finding the system for large positive times in a state, contained in a second subspace $\mathscr{H}_+$, where $P_+$, the projector upon $\mathscr{H}_+$, also commutes with $U_0(t, s)$ for every $t, s$. Then

$$W(P_-, \psi^{\text{in}}, P_+) = \lim_{t \to +\infty} \langle \psi(t) | P_+ \psi(t) \rangle = \langle \Psi^{\text{out}} | P_+ \psi^{\text{out}} \rangle$$
$$= \langle U_0(0, s)\psi^{\text{in}} | S^*(0) P_+ S(0) U_0(0, s)\psi^{\text{in}} \rangle, \tag{4.8}$$

where (4.6*b*) has been used.

To relate the scattering operator to a cross section the usual approach in the theory of potential scattering consists of choosing special projectors $P_\pm$, related to cones in relative momentum space. Afterwards the limit is taken where the aperture of the cone associated with $P_+$ goes to zero and where the overall effect of a large number of individual two-particle scattering processes is considered. This procedure then results in the standard relation between scattering operator and differential cross section (see Prugovečki 1971, chap 5 for details). We shall not try to perform this program for the case at hand but rather we shall point out a few features, specific for the present situation.

(i) Let us first turn our attention to the case considered in § 3, where the field is localized in space. Thus we can imagine a situation where the production apparatus for the particles as well as the detection system are well outside the field region. Hence we may assume that the particles are produced in a well defined internal state, let us say the eigenstate $\alpha$ of $K^{\text{int}}$, and that the detector measures the number of particles in a second eigenstate $\beta$ of $K^{\text{int}}$. Thus we are naturally led to projection operators $P_\pm$ of the type

$$P_- = P_-^{\text{tr}} \otimes P_\alpha, \qquad P_+ = P_+^{\text{tr}} \otimes P_\beta, \tag{4.9}$$

where $P_\alpha$ and $P_\beta$ are the projectors upon the internal states $\alpha$ and $\beta$, respectively. Obviously $P_\pm$ commute with $U_0(t, s) = U_0^{\text{tr}}(t, s) \otimes U_0^{\text{int}}(t, s)$, provided $P_\pm^{\text{tr}}$ commute with its translational part. It is also clear, from the relation $U_0^{\text{int}}(t, s)P_\gamma = \exp[-\mathrm{i}\omega_\gamma(t-s)]P_\gamma$ for $\gamma = \alpha, \beta$, that the internal parts of the free evolution operator, occurring in (4.8), cancel in the present situation. For the translational motion we can again consider the scattering from cones into cones. There is, however, the complication that the centre-of-mass motion is not decoupled from the motion of the rest of the system. Thus we have to consider separate cones for each of the two colliding particles. (Obviously, depending upon the initial preparation of the system, the two particles may collide within the field region but the collision process may also take place outside this area.) In fact the situation is somewhat reminiscent of a three-particle scattering process, once the centre-of-mass motion has been decoupled in that case.

Apart from the above complications, there seem to be no further problems in obtaining the desired expression for the differential cross section for scattering from internal states $\alpha$ into internal states $\beta$. We note further that at the end of the calculation the translational parts of $U_0(0, s)$ also disappear from the end result, ie $S(s)$ leads to the same cross section for each value of $s$. (This can be seen heuristically by substituting plane-wave eigenstates of $K^{\text{tr}}$ for the translational part of $\psi^{\text{in}}$ in relation (4.8).)

We close this part of the discussion by remarking that the present result refers to the situation where the two particles are outside the field region for both large positive and negative times. A different 'channel' may occur if initially one particle is trapped inside this area. This situation was not considered in § 3.

(ii) Next we consider the situation encountered in § 2, where the external field extends over all space. Although the asymptotic translational motion is not affected by the field and the centre-of-mass motion can be decoupled due to the assumed homogeneity of the field, we face the problem that the internal motion of the particles is influenced by the field in the production and detection areas also. Now we can imagine a detection system which measures the number of particles with a certain momentum, irrespective of their internal states. (In the absence of a field this would result in the measurement of a differential cross section, summed over the final internal states.) This would lead to a $P_+$ of the form $P_+ = P_+^{\text{tr}} \otimes I^{\text{int}}$, where $P_+^{\text{tr}}$ is of the standard type. The real problem occurs at the production end. The question is whether we can produce particles in a well defined internal state under the present circumstances and, if so, whether the projector upon this state commutes with $U_0(t, s)$ for every $t$ and $s$. It may of course happen that there are no one-dimensional subspaces of $\mathscr{H}^{\text{int}}$ which are left invariant by $U_0^{\text{int}}(t, s)$. In this case we can consider in-states which are mixed, ie density operators $\rho^{\text{in}}$. If the number $n_1 + n_2$ of internal states is finite, we can consider a situation where all pure states are equally probable initially, so that $\rho^{\text{in}} = (n_1 + n_2)^{-1} I^{\text{int}}$ and $P_- = P_-^{\text{tr}} \otimes I^{\text{int}}$ commutes with $U_0(t, s)$, provided this is the case for $P_-^{\text{tr}}$ and $U_0^{\text{tr}}(t, s)$. (In practice, for a two-level system an equal distribution over the two states with energy difference $\hbar\Delta\omega$ may be achieved by means of a strong resonant field in the production region which causes saturation.)

Obviously one can avoid the penetration of the time-dependent field into the asymptotic regions in actual scattering experiments. The situation of § 2 is relevant, however, in a number of statistical-mechanical problems. There the existence of the wave operators is quite important, but, since the results of individual scattering processes are not monitored, this is not so for the concept of cross section.

## Acknowledgments

EP was supported in part by a grant from the National Research Council of Canada. AT was supported by the Foundation for Fundamental Research on Matter (FOM), which is sponsored by the Netherlands Organization for the Advancement of Pure Research (ZWO).

## References

Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin, Heidelberg and New York: Springer)
Prugovečki E 1971 *Quantum Mechanics in Hilbert Space* (New York and London: Academic Press)
Prugovečki E and Tip A 1974 *J. Phys.* A: *Math. Nucl. Gen.* **7** 572–85
Prugovečki E and Zorbas J 1973 *J. math. Phys.* **14** 1398–409